2.8
operations on divergent Sequences.

2.8A : Theorem: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers that diverge to infinity then $\{s_n+t_n\}_{n=1}^{\infty}$ and $\{s_n t_n\}_{n=1}^{\infty}$ diverges to infinity.

Proof: Given the sequence $\{s_n\}_{n=1}^{\infty}$ diverge to infinity

$\therefore$ By definition given $M>0$ there exists $N_1 \in I$ such that $s_n > M \quad \forall\, n \geq N_1$ —— ①

Also given $\{t_n\}_{n=1}^{\infty}$ diverges to infinity, there exists $N_2 \in I$ such that $t_n > 1 \quad \forall\, n \geq N_2$ —— ②

Let $N = \max\{N_1, N_2\}$

$\forall\, n \geq N$ using ① & ②

$s_n + t_n > M+1 \quad \forall\, n \geq N \Rightarrow \{s_n+t_n\}_{n=1}^{\infty}$ diverges to infinity

and $s_n t_n > M \quad \forall\, n \geq N$

$\Rightarrow \{s_n t_n\}_{n=1}^{\infty}$ diverges to ∞.

Hence proved.

2.8B. Theorem: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\{s_n\}_{n=1}^{\infty}$ diverges to infinity, and if $\{t_n\}_{n=1}^{\infty}$ is bounded, then $\{s_n+t_n\}_{n=1}^{\infty}$ diverges to infinity.

Proof: Given $\{t_n\}_{n=1}^{\infty}$ is bounded sequence

$\therefore$ By defn there exists $Q>0$ such that

$|t_n| \leq Q \quad \forall\, n \in I$ —— ①

Also given $\{s_n\}_{n=1}^{\infty}$ diverge to infinity then there exists $N \in I$ ~~such~~ given $M > 0$ such that $s_n > M + Q \quad \forall\, n \geq N$.

Then for $n \geq N$

$$s_n + t_n > s_n - |t_n| > M + Q - Q$$

$$\Rightarrow \quad s_n + t_n > M \quad \forall\, n \geq N$$

$\Rightarrow$ seq $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to infinity

Hence proved.

Theorem 2.8C: If $\{s_n\}_{n=1}^{\infty}$ diverges to infinity and if $\{t_n\}_{n=1}^{\infty}$ converges, then $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to infinity.

Proof: Given $\{t_n\}_{n=1}^{\infty}$ converges.

Every convergent sequence is bounded sequence.

$\therefore \{t_n\}_{n=1}^{\infty}$ is bounded sequence

Also given $\{s_n\}_{n=1}^{\infty}$ diverges to infinity

$\therefore$ By theorem 2.8 B

$\{s_n + t_n\}_{n=1}^{\infty}$ is diverges to infinity.

Hence proved.

2.9 Limit superior and limit inferior.

Let us consider a sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above and let $M_1 = l.u.b \{s_1, s_2, s_3, \ldots\}$

$$M_2 = l.u.b \{s_2, s_3, s_4, \ldots\}$$

$$M_3 = l.u.b \{s_3, s_4, s_5 \ldots\}$$

$$\vdots$$

$$M_n = l.u.b \{s_n, s_{n+1}, s_{n+2}, \ldots\}$$

Clearly $M_1 \geq M_2 \geq M_3 \geq \cdots M_n \geq M_{n+1} \geq \cdots$

ie) The sequence $\{M_n\}_{n=1}^{\infty}$ is decreasing sequence.

2.9A Definition : Limit superior.

Let $\{s_n\}_{n=1}^{\infty}$ be sequence of real numbers that is bounded above, and let $M_n = \text{l.u.b}\{s_n, s_{n+1}, s_{n+2}, \cdots\}$

a) If $\{M_n\}_{n=1}^{\infty}$ converges, we define $\lim\limits_{n\to\infty} M_n = \limsup\limits_{n\to\infty} s_n$

b) If $\{M_n\}_{n=1}^{\infty}$ diverges to minus infinity, we write

$$\limsup_{n\to\infty} s_n = -\infty.$$

2.9B Defn : If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers that is not bounded above ~~the~~ we write $\limsup\limits_{n\to\infty} s_n = \infty$

2.9C Theorem :

If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers, then $\limsup\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} s_n$.

Proof Given $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence.

$$\therefore \lim_{n\to\infty} s_n = L \ \text{(say)}$$

$\Rightarrow$ given $\varepsilon > 0$ there exists $N \in I$ such that

$$|s_n - L| < \varepsilon \quad \forall n \geq N$$

$$-\varepsilon < s_n - L < \varepsilon$$

$$L - \varepsilon < s_n < L + \varepsilon \quad \forall n \geq N.$$

Thus if $n \geq N$, then $L+\varepsilon$ is an upper bound for $\{s_n, s_{n+1}, s_{n+2} \cdots\}$ and $L-\varepsilon$ is not an upper bound

Hence $L-\varepsilon_1 < M_n = \text{l.u.b}\{s_n, s_{n+1}, \cdots\} \leq L+\varepsilon_1$

$$\Rightarrow \quad L-\varepsilon_1 \leq \lim_{n\to\infty} M_n \leq L+\varepsilon_1$$

$$L-\varepsilon_1 \leq \limsup_{n\to\infty} s_n \leq L+\varepsilon_1$$

$$\Rightarrow \limsup_{n\to\infty} s_n = L.$$

Defn If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ is bounded below, then the set $\{s_n, s_{n+1}, s_{n+2}, \cdots\}$ has a g.l.b.

Let $m_n = \text{g.l.b}\{s_n, s_{n+1}, s_{n+2}, \cdots\}$, then $\{m_n\}_{n=1}^{\infty}$ is a nondecreasing sequence (increasing)

Defn Limit inferior

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real number that is bounded below and let $m_n = \text{g.l.b}\{s_n, s_{n+1}, s_{n+2}, \cdots\}$

a) If $\{m_n\}_{n=1}^{\infty}$ converges, we define.

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} m_n .$$

b) If $\{m_n\}_{n=1}^{\infty}$ diverges to ∞, we write $\liminf_{n\to\infty} s_n = \infty$

2.9E **Defn** If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real number that is not bounded below, we write $\liminf_{n\to\infty} s_n = -\infty$

2.9F Theorem: If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers, then $\lim_{n\to\infty} \text{Inf}\, s_n = \lim_{n\to\infty} s_n$.

Proof: Given $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence.

(ie) $\lim_{n\to\infty} s_n = L$

By defn, given $\epsilon > 0$ there exists $N \in I$ such that

$$|s_n - L| < \epsilon \quad \forall\, n \geq N$$

$$-\epsilon < s_n - L < \epsilon \quad \forall\, n \geq N$$

$$L - \epsilon < s_n < L + \epsilon \quad \forall\, n \geq N$$

Thus, if $n \geq N$, then $L - \epsilon$ is l.b for $\{s_n, s_{n+1}, s_{n+2}, \ldots\}$ and $L + \epsilon$ is not an lower bound.

$$L - \epsilon < \text{g.l.b}\{s_n, s_{n+1}, s_{n+2}, \ldots\} < L + \epsilon \quad \forall\, n \geq N$$

$$L - \epsilon < m_n < L + \epsilon \quad \forall\, n \geq N$$

$$|m_n - L| < \epsilon \quad \forall\, n \geq N$$

$$\lim_{n\to\infty} m_n = L$$

$$\Rightarrow \lim_{n\to\infty} \text{Inf}\, s_n = L$$

$$\Rightarrow \lim_{n\to\infty} \text{Inf}\, s_n = \lim_{n\to\infty} s_n$$

Hence proved.

If $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} \sup s_n = \lim \text{Inf}\, s_n = L$

Proof: Both 2.9E and 2.9F

Theorem 2.9 G If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers then $\lim\limits_{n\to\infty} \text{Inf}\, s_n \leq \lim\limits_{n\to\infty} \text{Sup}\, s_n$.

Proof : Case (i) If sequence $\{s_n\}_{n=1}^{\infty}$ is bounded, then

$$m_n = g.l.b\{s_n, s_{n+1}, s_{n+2} \cdots\} \leq l.u.b\{s_n, s_{n+1}, s_{n+2} \cdots\} = M_n$$

$$m_n \leq M_n$$

$$\lim_{n\to\infty} m_n \leq \lim_{n\to\infty} M_n$$

$$\lim_{n\to\infty} \text{Inf}\, s_n \leq \lim_{n\to\infty} \text{sup}\, s_n$$

Case (ii) If $\{s_n\}_{n=1}^{\infty}$ is not bounded, then either $\lim\limits_{n\to\infty} \text{sup}\, s_n = \infty$ and $\lim \text{Inf}\, s_n = -\infty$

$$\Rightarrow \lim_{n\to\infty} \text{Inf}\, s_n \leq \lim_{n\to\infty} \text{sup}\, s_n$$

Hence proved.

Theorem : 2.9H.

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and if $\lim\limits_{n\to\infty} \text{sup}\, s_n = \lim\limits_{n\to\infty} \text{Inf}\, s_n = L$ where $L \in R$, then $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} s_n = L$.

Proof : Given $\lim\limits_{n\to\infty} \text{Sup}\, s_n = L$

$$\lim_{n\to\infty} M_n = L$$

$$\lim_{n\to\infty} l.u.b\{s_n, s_{n+1}, s_{n+2} \cdots\} = L$$

By defn given $\epsilon > 0$ $\exists$ $N_1 \in I$ such that

$$\left| l.u.b \{s_n, s_{n+1}, s_{n+2} \cdots\} - L \right| < \varepsilon_1 \quad \forall\, n \geq N_1$$

$$L - \varepsilon_1 < l.u.b \{s_n, s_{n+1} \cdots\} < L + \varepsilon_1$$

$$\Rightarrow \quad \cancel{s_n - L < \varepsilon_1 \quad \forall\, n \geq N_1}$$

$$s_n < L + \varepsilon_1$$

$$s_n < L + \varepsilon_1 \quad \forall\, n \geq N_1 \quad \text{—— (1)}$$

similarly since $\lim\limits_{n\to\infty} \operatorname{Inf} s_n = L$, there exists $N_2 \in I$

such that $\cancel{\lim} \left| g.l.b \{s_n, s_{n+1}, s_{n+2}\} - L \right| < \varepsilon_1 \quad \forall\, n \geq N_2$

$$L - \varepsilon_1 < g.l.b \{s_n, s_{n+1}, s_{n+2} \cdots\} < L + \varepsilon_1$$

$$\forall\, n \geq N_2 \quad \text{—— (2)}$$

$$L - \varepsilon_1 < s_n$$

Let $N = \max\{N_1, N_2\}$

$\Rightarrow \forall\, n \geq N$

$$L - \varepsilon_1 < s_n < L + \varepsilon_1$$

$$\Rightarrow |s_n - L| < \varepsilon_1 \quad \forall\, n \geq N$$

$$\Rightarrow \lim_{n\to\infty} s_n = L$$

Hence proved.